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Ends, fundamental tones, and capacities of minimal submanifolds via extrinsic comparison theory

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Abstract We study the volume of extrinsic balls and the capacity of extrinsic annuli in minimal submanifolds which are properly immersed with controlled radial sectional curvatures into an ambient manifold with a pole. The key results are concerned with the comparison of those volumes and capacities with the corresponding entities in a rotationally symmetric model manifold. Using the asymptotic behavior of the volumes and capacities we then obtain upper bounds for the number of ends as well as estimates for the fundamental tone of the submanifolds in question.

Keywords First Dirichlet eigenvalue · Capacity · Effective resistance · Minimal submanifolds · Fundamental tone · minimal submanifolds

Mathematics Subject Classification (2000) 53A · 53C

1 Introduction

Let M be a complete non-compact Riemannian manifold. Let $K \subset M$ be a compact set with non-empty interior and smooth boundary. We denote by $\mathcal{E}_K(M)$ the number of connected components $E_1, \cdots, E_{\mathcal{E}_K(M)}$ of $M \setminus K$ with non-compact closure. Then M has $\mathcal{E}_K(M)$ ends $\{E_i\}_{i=1}^{\mathcal{E}_K(M)}$ with respect to K

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(see e.g. [14]), and the *global* number of ends $\mathcal{E}(M)$ is given by

$$\mathcal{E}(M) = \sup_{K \subset M} \mathcal{E}_K(M) \quad , \tag{1}$$

where K ranges on the compact sets of M with non-empty interior and smooth boundary.

The number of ends of a manifold can be bounded by geometric restrictions. For example, in the particular setting of an $m-$dimensional minimal submanifold P which is properly immersed into Euclidean space $\mathbb{R}^n$, the number of ends $\mathcal{E}(P)$ is known to be related to the extrinsic properties of the immersion. Indeed, V. G. Tkachev proved in theorem 2 of [27] (see also [5]) that for any properly immersed $m-$dimensional minimal submanifold P in $\mathbb{R}^n$ with finite volume growth ($V_{w_0}(P) < \infty$) the number of ends is bounded from above by

$$\mathcal{E}(P) \leq C_m V_{w_0}(P) \quad , \tag{2}$$

where $C_m = 1$ ($C_m = 2^m$ in the original [27]) and the volume growth $V_{w_0}(P)$ is

$$V_{w_0}(P) = \lim_{R\to\infty} \frac{\operatorname{Vol}\left(P \cap B_R^{\mathbb{R}^n}(o)\right)}{\operatorname{Vol}\left(B_R^{\mathbb{R}^m}(o)\right)} \quad . \tag{3}$$

Here $\operatorname{Vol}\left(B_R^{\mathbb{R}^m}(o)\right)$ is the volume of a geodesic ball $B_R^{\mathbb{R}^m}(o)$ of radius R centered at o in $\mathbb{R}^m$. The inequality (2) thus shows a significant relation between the number of ends (*i.e.* a topological property) and the behavior of a quotient of volumes (*i.e.* a metric property).

Motivated by Tkachev's application of the *volume quotient* appearing in equation (3), we will consider the corresponding *flux quotient* and *capacity quotient* of the minimal submanifolds. These quotients are constructed in the same way as indicated by the *volume quotient* but here we generalize the setting as well as Tkachev's result to minimal submanifolds in more general ambient spaces as alluded to in the abstract. Specifically we assume that the minimal immersion goes into an ambient manifold N with a pole and with sectional curvatures K_N bounded from above by the radial curvatures K_w of a rotationally symmetric model space $M_w^n = \mathbb{R}^+ \times \mathbb{S}_1^{n-1}$, with warped metric tensor $g_{M_w^n}$ constructed using a positive warping function $w : \mathbb{R}^+ \to \mathbb{R}^+$ in such a way that $g_{M_w^n} = dr^2 + w(r)^2 g_{\mathbb{S}_1^{n-1}}$ is also balanced from below (see [22] and §3 for precise definitions).

Our generalization of inequality (2) is thence the following:

Theorem 1 *Let $\varphi : P^m \to N^n$ be a proper minimal and complete immersion into an $n-$dimensional ambient manifold N^n which possesses a pole $o \in N^n$ and have its sectional curvatures K_N at any point $p \in N$ bounded from above by the radial curvatures K_w of a model space M_w^n (which itself is assumed to be balanced from below):*

$$K_N(p) \leq K_{M_w^n}(r(p)) = -\frac{w''}{w}(r(p)) \quad . \tag{4}$$

Suppose moreover, that $w' > 0$ and there exists R_0 such that $K_{M^n_w}(R) \leq 0$ for any $R > R_0$. Then, the number of ends $\mathcal{E}_{D_R}(P)$ with respect to the extrinsic ball $D_R = P \cap B^N_R(o)$ for $R > R_0$ is bounded from above by

$$\mathcal{E}_{D_R}(P) \leq \left(\frac{2}{1-\frac{R}{t}}\right)^m \left(\frac{\int_0^t w(s)^{m-1}ds}{t^m/m}\right) \frac{\operatorname{Vol}(D_t)}{\operatorname{Vol}(B^w_t)} \quad , \tag{5}$$

for any $t > R$.

Using the above theorem we can estimate the global number of ends as follows:

Corollary 1 *Under the assumptions of theorem 1, suppose moreover*

$$\limsup_{t\to\infty} \left(\frac{\int_0^t w(s)^{m-1}ds}{t^m/m}\right) = C_w < \infty \quad , \tag{6}$$

and suppose also that the submanifold has finite volume growth, namely

$$\operatorname{Vol}_w(P) = \lim_{t\to\infty} \frac{\operatorname{Vol}(D_t)}{\operatorname{Vol}(B^w_t)} < \infty \quad . \tag{7}$$

Then

$$\mathcal{E}(P) \leq 2^m C_w \operatorname{Vol}_w(P) \quad . \tag{8}$$

Remark 1 If we choose $w(t) = w_0(t) = t$, the model space becomes $\mathbb{R}^m$, which is balanced from below, and the hypothesis of theorem 1 are therefore automatically fulfilled for any complete minimal submanifold properly immersed in a Cartan-Hadamard ambient manifold. Inequality (5) becomes

$$\mathcal{E}_{D_R}(P) \leq \left(\frac{2}{1-\frac{R}{t}}\right)^m \frac{\operatorname{Vol}(D_t)}{V_m t^m} \quad , \tag{9}$$

For any $R > 0$ and any $t > R$, where V_m denotes the volume of a geodesic ball of radius 1 in $\mathbb{R}^m$. From inequality (6) we get

$$C_{w_0} = 1 \quad . \tag{10}$$

Thus inequality (8) becomes

$$\mathcal{E}(P) \leq 2^m \lim_{t\to\infty} \frac{\operatorname{Vol}(D_t)}{V_m t^m} \quad , \tag{11}$$

which is the original inequality obtained by Tkachev (inequality (2)), but now inequality (11) is valid for any minimal submanifold properly immersed in a Cartan-Hadamard ambient manifold with finite volume growth.

In [10, 5] are also obtained lower bounds for the number of ends, but we note that those lower bounds seem to need stronger assumptions: Dimension greater than 2, or embeddedness of the ends and codimension 1, decay on the second fundamental form, and a rotationally symmetric ambient manifold. As a counterpart, those lower bounds are associated to the so-called gap type theorems.

Combining the results of [10, Theorem 3.5] and corollary 1, and taking into account the role of sectional curvatures of the model space (see [10, Proposition 2.6]) we have

Corollary 2 *Let $\varphi : P^m \to M^n_w$ be a minimal and proper immersion into a model space M^n_w which is balanced from below with an increasing warping function w satisfying the following conditions:*

$$\begin{aligned}
&\limsup_{t\to\infty}\left(\frac{\int_0^t w(s)^{m-1}ds}{t^m/m}\right) = C_w < \infty \quad ,\\
&\text{there exists } R_0 \text{ such that for any } R > R_0\\
&\begin{cases} -\frac{w''(R)}{w(R)} \leq 0 \\ \frac{1-\left(w'(R)\right)^2}{w(R)^2} \leq 0 \end{cases}
\end{aligned} \tag{12}$$

Suppose moreover $m > 2$, the center o_w of M^n_w satisfies $\varphi^{-1}(o_w) \neq \emptyset$ and the norm of the second fundamental form $\|B^P\|$ of the immersion is bounded for large r by

$$\|B^P\| \leq \frac{\epsilon(r)}{w'(r)w(r)} \quad , \tag{13}$$

where ϵ is a positive function such that $\epsilon(r) \to 0$ when $r \to \infty$.

Then the number of ends is bounded from below and from above as follows

$$\mathrm{Vol}_w(P) \leq \mathcal{E}(P) \leq 2^m C_w \,\mathrm{Vol}_w(P) \quad . \tag{14}$$

By using our results about the behavior of the comparison quotients we can also estimate the capacity of an extrinsic annulus $\mathrm{A}_{\rho,R} = P \cap \left(B^N_R(o) \setminus B^N_\rho(o)\right)$ (see figure 1, §2, and §3 for a precise definition of *capacity* and *extrinsic annulus*):

Theorem 2 *Let $\varphi : P^m \to \mathbb{R}^n$ denote a complete and proper minimal immersion into the Euclidean space $\mathbb{R}^n$. Then, for any $R > \rho > 0$, the capacity of the extrinsic annulus $A_{\rho,R}$ is bounded from below and from above as follows:*

$$\frac{\mathrm{Vol}(D_\rho)}{V_m \rho^m} \leq \frac{\mathrm{Cap}(A_{\rho,R})}{\mathrm{Cap}(A^{\mathbb{R}^m}_{\rho,R})} \leq \frac{\mathrm{Vol}(D_R)}{V_m R^m} \quad , \tag{15}$$

where $\mathrm{Cap}(A^{\mathbb{R}^m}_{\rho,R})$ is the capacity of the geodesic annulus $A^{\mathbb{R}^m}_{\rho,R}$ in $\mathbb{R}^m$ of inner radius ρ and outer radius R.

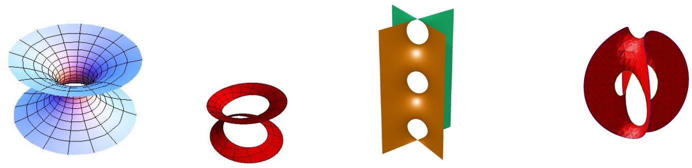

Fig. 1 Two examples of extrinsic annuli in $\mathbb{R}^3$: A catenoid on the left and the singly periodic Scherk surface on the right. The extrinsic annuli are constructed by cutting the surfaces with two spheres (with the same center but of different radii) in the ambient manifold ($\mathbb{R}^3$). The catenoid has two ends and finite total curvature. Hence, by theorem 2, the capacity of the extrinsic annulus of the catenoid is greater than the capacity of the corresponding annulus of the Euclidean 2-plane but is smaller than two times that capacity. The same is true for the extrinsic annulus of the singly periodic Scherk surface (we refer the reader to the introduction of [24] for the area growth of the singly periodic Scherk surface).

Remark 2 Since, from Theorem 3, the quotient $\frac{\mathrm{Vol}(D_s)}{V_m s^m}$ is a non-decreasing function of s, we can state Theorem 2 in the limit case ($\rho \to 0$ and $R \to \infty$) and inequality (15) there becomes

$$1 \leq \frac{\mathrm{Cap}(A_{\rho,R})}{\mathrm{Cap}(A^{\mathbb{R}^m}_{\rho,R})} \leq \lim_{R\to\infty} \frac{\mathrm{Vol}(D_R)}{V_m R^m} = V_{w_0}(P) \quad . \tag{16}$$

When we deal with a minimal surface $\Sigma \subset \mathbb{R}^3$ which is properly *embedded* into the Euclidean space $\mathbb{R}^3$ the limit

$$V_{w_0}(\Sigma) = \lim_{R\to\infty} \frac{\mathrm{Vol}(\Sigma \cap B^{\mathbb{R}^3}_R(o))}{\pi R^2} \tag{17}$$

is well understood. For instance, the above limit corresponds to the number of ends if the surface Σ has finite total curvature.

Remark 3 In order to bound the capacity quotient, our theorems do not make use of the volume quotient as in Theorem 2, but instead they make use of the flux quotient (see Theorems 4 and 5). In the special case when the ambient manifold is $\mathbb{R}^n$ (such as in Theorem 2) the volume quotient agrees however with the flux quotient (see equation (105) and theorem 6).

1.1 Outline of the paper

In §2 we show our main theorems concerning the flux quotients, the volume quotients, and the capacity quotients. In §3 we state the preliminary concepts in order to prove the main theorems of §2 in §4. This allows us then to prove Theorem 1 and Corollary 1 in §5. Finally, in §6, we present several corollaries and examples of applications of the extrinsic theory and results which have been established in §2.

2 Extrinsic theory: Flux, Capacity and Volume comparison for extrinsic balls

Let (M^n, g) be a Riemannian manifold. For any oriented hypersurface $\Sigma \subset M$ with unit normal vector field ν, we define the flux $F_X(\Sigma)$ of the vector field X through Σ by

$$F_X(\Sigma) := \int_\Sigma \langle X, \nu\rangle d\mu_\Sigma \quad , \tag{18}$$

where $d\mu_\Sigma$ is the associated Riemannian density determined by the metric $g_\Sigma = i^*g$ (where $i : \Sigma \to M$ denotes the inclusion map).

By the divergence theorem (see [4] for instance), if one has an oriented domain Ω in M with smooth boundary $\partial\Omega$, and the vector field X is C^1 in $\overline{\Omega}$ and with compact support in $\overline{\Omega}$, the flux of X through $\partial\Omega$ is related to the divergence of X by

$$\int_\Omega \operatorname{div} X d\mu = \int_{\partial\Omega} \langle X, \nu\rangle d\mu_{\partial\Omega} = F_X(\partial\Omega) \quad . \tag{19}$$

Given a smooth function $u : M \to \mathbb{R}$, we can also define the flux of a function u, but then the flux $J_u(t)$ is the flux of the gradient ∇u (*i.e.* the metric dual vector to du, $du(X) = \langle \nabla u, X\rangle$) through the level set $\Sigma_t^u := \{x \in M \,|\, u(x) = t\}$ so that:

$$J_u(t) := F_{\nabla u}(\Sigma_t^u) \quad . \tag{20}$$

Taking into account that the outward unit normal vector field ν of Σ_t^u is $\nu = \frac{\nabla u}{|\nabla u|}$, it is easy to see that

$$J_u(t) = \int_{\Sigma_t} |\nabla u| d\mu_{\Sigma_t^u} \quad . \tag{21}$$

Observe moreover, that by the Sard theorem and by the regular set theorem we need no further restrictions on the smoothness of Σ_t^u and on the smoothness of the unit normal vector field ν.

The overall goal of this work is to characterize the *isoperimetric inequalities for extrinsic balls*, and the *capacity of minimal submanifolds* in terms of the flux of extrinsic distance functions. Actually we are interested in the flux of the extrinsic distance function on minimal submanifolds in an ambient manifold N which possesses a pole and has the radial curvatures bounded from above by the radial curvatures of rotationally symmetric model space $K_N \leq K_{M_w^n} = -\frac{w''}{w}$, see [22] or section 3 of this paper for precise definitions. It is the behavior of this particular flux that allows us to study the mean exit time function, the capacity, the conformal type, the fundamental tone, and in special cases also the number of ends of the submanifold.

2.1 Flux and volume comparison: isoperimetric inequalities and the mean exit time function

Given an isometric immersion $\varphi : P \to (N, o)$ into a manifold with a pole $o \in N$, the flux J_r of the extrinsic distance function r_o (*i.e.*, the restriction by the immersion of the ambient distance function to the submanifold) is given by

$$J_r(R) = \int_{\partial D_R} |\nabla^P r_o| d\mu \quad ,$$

where ∂D_R is the level set $\partial D_R = r_o^{-1}(R)$, and therefore, $D_R = r_o^{-1}([0, R))$ is the extrinsic ball of radius R.

When the immersion is minimal and the ambient manifold has its radial sectional curvatures K_N bounded from above by the radial sectional curvatures of a rotationally symmetric model space M_w^n that is balanced from below (see [22] and section 3), $K_N \leq K_{M_w^n}$, we can compare the volume quotient $\frac{\mathrm{Vol}(D_R)}{\mathrm{Vol}(B_R^w)}$ and the flux quotient $\frac{J_r(R)}{J_r^w(R)}$. The volume quotient is the quotient between the volume of a extrinsic ball D_R of radius R in P^m and the volume of a geodesic ball B_R^w of the same radius R in M_w^m. The flux quotient is the quotient between the flux of the extrinsic distance in P^m and the flux of the geodesic distance in M_w^m. These two quotients are related by the following theorem

Theorem 3 *Let $\varphi : P^m \longrightarrow N^n$ be an isometric, proper, and minimal immersion of a complete non-compact Riemannian m-manifold P^m into a complete Riemannian manifold N^n with a pole $o \in N$. Let us suppose that the $o-$radial sectional curvatures of N are bounded from above by*

$$K_{o,N}(\sigma_x) \leq -\frac{w''(r)}{w(r)}(\varphi(x)) \;\; \forall x \in P \, ,$$

and that the model space M_w^m is balanced from below. Then

1. *$J_r(R)$ is related with $\mathrm{Vol}(D_R)$ by*
$$\frac{\mathrm{Vol}(D_R)}{\mathrm{Vol}(B_R^w)} \leq \frac{J_r(R)}{J_r^w(R)}. \tag{22}$$
2. *The functions $\frac{\mathrm{Vol}(D_R)}{\mathrm{Vol}(B_R^w)}$ and $\frac{J_r(R)}{J_r^w(R)}$ are non decreasing functions of R.*
3. *Denoting by $E_R^P(x)$ the mean time for the first exit from the extrinsic ball $D_R(o)$ for a Brownnian particle starting at $o \in P^m$, and denoting by E_R^w the mean exit time function for the $R-$ball B_R^w in the model space M_w^m, if equality holds in (22) for some fixed radius $R > 0$, then for any $x \in D_R$, $E_R^P(x) = E_R^w(r(x))$, where $r(x)$ the extrinsic distance from o to the point $x \in P$.*

2.2 Capacity and flux comparison: conformal type

Given a compact set $F \subset M$ in a Riemannian manifold M and an open set $G \subset M$ containing F, we call the couple (F, G) a *capacitor*. Each capacitor then has its capacity defined by

$$\mathrm{Cap}(F, G) := \inf_u \int_{G \setminus F} \|\nabla u\|^2 d\mu \quad , \tag{23}$$

where the inf is taken over all Lipschitz functions u with compact support in G such that $u = 1$ on F.

When G is precompact, the infimum is attained for the function $u = \Psi$ which is the solution of the following Dirichlet problem in $G - F$:

$$\begin{cases} \Delta\Psi = 0 \\ \Psi|_{\partial F} = 1 \\ \Psi|_{\partial G} = 0 \end{cases} \tag{24}$$

Remark 4 Observe that we can define a rescaled function Ψ' from the above function Ψ such that

$$\Psi' = -\Psi + 1.$$

Hence, Ψ' is the smooth function which satisfies

$$\begin{cases} \Delta\Psi' = 0 \\ \Psi'|_{\partial F} = 0 \\ \Psi'|_{\partial G} = 1 \end{cases} \tag{25}$$

and therefore,

$$\mathrm{Cap}(F, G) = \int_{G \setminus F} \|\nabla \Psi'\|^2 d\mu = \int_{G \setminus F} \|\nabla \Psi\|^2 d\mu \quad . \tag{26}$$

From a physical point of view, the capacity of the capacitor (F, G) represents the total electric charge (generated by the electrostatic potential Ψ) flowing into the domain $G - F$ through the interior boundary ∂F. Since the total current stems from a potential difference of 1 between ∂F and ∂G, we get from Ohm's Law that the effective resistance of the domain $G - F$ is

$$R_{\mathrm{eff}}(G - F) = \frac{1}{\mathrm{Cap}(F, G)} \quad . \tag{27}$$

The exact value of the capacity of a set is known only in a few cases, and so its estimation in geometrical terms is of great interest, not only in electrostatic, but in many physical descriptions of flows, fluids, heat, or generally where the Laplace operator plays a key role, see [7, 15].

Given a capacitor (F, G), if we have a smooth function u with $u = a$ on ∂F and $u = b$ on ∂G, then the capacity and the flux are then related by (see [13]):

$$\mathrm{Cap}(F, G) \leq \left(\int_a^b \frac{ds}{J_u(s)} \right)^{-1} \quad . \tag{28}$$

In this paper we are interested on the o-centered *extrinsic annulus* $A_{\rho,R}(o)$ for $0 < \rho < R$ given by

$$A_{\rho,R}(o) := D_R(o) - D_\rho(o) \quad . \tag{29}$$

To be more precise, we are interested on the behavior of the flux and the capacity of those extrinsic domains. In the following theorems we provide upper and lower bounds for the capacity quotient in terms of the flux quotient.

Theorem 4 *Let $\varphi : P^m \longrightarrow N^n$ be an isometric, proper, and minimal immersion of a complete non-compact Riemannian m-manifold P^m into a complete Riemannian manifold N^n with a pole $o \in N$ and satisfying $\varphi^{-1}(o) \neq \emptyset$. Let us suppose that the $o-$radial sectional curvatures of N are bounded from above by*

$$K_{o,N}(\sigma_x) \leq -\frac{w''(r)}{w(r)}(\varphi(x)) \;\; \forall x \in P \quad ,$$

and that the warping function w satisfies

$$w' \geq 0 \quad .$$

Then

$$\frac{J_r(\rho)}{J_r^w(\rho)} \leq \frac{\mathrm{Cap}(A_{\rho,R})}{\mathrm{Cap}(A_{\rho,R}^w)} \quad , \tag{30}$$

where $A_{\rho,R}^w$ is the intrinsic annulus in M_w^m.

Theorem 5 *Let $\varphi : P^m \longrightarrow N^n$ be an isometric, proper, and minimal immersion of a complete non-compact Riemannian m-manifold P^m into a complete Riemannian manifold N^n with a pole $o \in N$. Let us suppose that the $o-$radial sectional curvatures of N are bounded from above by*

$$K_{o,N}(\sigma_x) \leq -\frac{w''(r)}{w(r)}(\varphi(x)) \;\; \forall x \in P \,,$$

and that the model space M_w^m is balanced from below. Then

$$\frac{\mathrm{Cap}(A_{\rho,R})}{\mathrm{Cap}(A_{\rho,R}^w)} \leq \frac{J_r(R)}{J_r^w(R)} \quad , \tag{31}$$

where $A_{\rho,R}^w$ is the intrinsic annulus in M_w^m. Moreover, if equality holds in (31) for some fixed $R > 0$, then D_R is a minimal cone in N^n.

Geometric estimates of the capacity are sufficient to obtain large scale consequences such as as the parabolic or hyperbolic character of the manifold, [17, 16, 20, 21]. We note here the following important equivalent conditions about the conformal type:

Theorem A *Let (M, g) be a given Riemannian manifold. Then the following conditions are equivalent:*

- *There is a precompact open domain K in M, such that the Brownian motion X_t starting from K does not return to K with probability* 1, i.e.,

$$P_x\left\{\omega | X_t(\omega) \in K \text{ for some } t > 0\right\} < 1 \quad . \tag{32}$$

- *M has positive capacity: There exists in M a compact domain K, such that*

$$\mathrm{Cap}(K, M) > 0 \quad . \tag{33}$$

- *M has finite resistance to infinity: There exists in M a compact domain K, such that*

$$R_{\mathit{eff}}(M - K) < \infty \quad . \tag{34}$$

A manifold satisfying the conditions of the above theorem will be called a hyperbolic manifold, otherwise it is called a parabolic manifold.

As a consequence of the above theorem we can state the following corollary for minimal submanifolds:

Corollary 3 *Let $\varphi : P^m \longrightarrow N^n$ be an isometric, proper, and minimal immersion of a complete non-compact Riemannian m-manifold P^m into a complete Riemannian manifold N^n with a pole $o \in N$. Let us suppose that the $o-$radial sectional curvatures of N are bounded from above by*

$$K_{o,N}(\sigma_x) \leq -\frac{w''(r)}{w(r)}(\varphi(x)) \;\; \forall x \in P \quad ,$$

and that the warping function w satisfies

$$w' \geq 0 \quad .$$

Then

1. *If M_w^m is a hyperbolic manifold, then P is a hyperbolic manifold.*
2. *In consequence, if P is parabolic, then M_w^m is also parabolic.*

Since $\frac{J_r(R)}{J_r^w(R)}$ and $\frac{\mathrm{Vol}(D_R)}{\mathrm{Vol}(B_R^w)}$ are non-decreasing functions under our hypothesis, we can define two expressions which are analogous to the projective volume defined by V. G. Tkachev in [27]

Definition 1 Given $\varphi : P^m \to N^n$ an immersion into a manifold N with a pole $o \in N$. The *w-flux* $\mathrm{Flux}_w(P)$ and the *w-volume* $\mathrm{Vol}_w(P)$ of the submanifold P are defined by :

$$\begin{aligned} \mathrm{Flux}_w(P) &:= \sup_{R\in\mathbb{R}^+} \frac{J_r(R)}{J_r^w(R)} \quad , \\ \mathrm{Vol}_w(P) &:= \sup_{R\in\mathbb{R}^+} \frac{\mathrm{Vol}(D_R)}{\mathrm{Vol}(B_R^w)} \quad . \end{aligned} \tag{35}$$

We will say that P has *finite $w-$ flux* (resp. *finite $w-$ volume*) if and only if $\mathrm{Flux}_w(P) < \infty$ (or $\mathrm{Vol}_w(P) < \infty$).

We refer to theorem 6 for the relation between the $w-$flux and the $w-$volume of a submanifold.

From theorem A and theorem 5 we can now state that for minimal submanifolds with finite $w-$flux we have:

Corollary 4 *Let $\varphi : P^m \longrightarrow N^n$ be an isometric, proper and minimal immersion of a complete non-compact Riemannian m-manifold P^m into a complete Riemannian manifold N^n with a pole $o \in N^n$. Let us suppose that the $o-$radial sectional curvatures of N^n are bounded from above as follows*

$$K_{o,N}(\sigma_x) \leq -\frac{w''(r)}{w(r)}(\varphi(x)) \;\; \forall x \in P \quad ,$$

and that the model space M_w^m is balanced from below. Suppose moreover that P has finite $w-$flux. Then

1. *If M_w^m is a parabolic manifold, then P is a parabolic manifold.*
2. *If P is an hyperbolic manifold, then M_w^n is an hyperbolic manifold.*

Joining the previous two corollaries together we get:

Corollary 5 *Let $\varphi : P^m \longrightarrow N^n$ be an isometric, proper and minimal immersion of a complete non-compact Riemannian m-manifold P^m into a complete Riemannian manifold N^n with a pole $o \in N^n$. Let us suppose that the $o-$radial sectional curvatures of N^n are bounded from above,*

$$K_{o,N}(\sigma_x) \leq -\frac{w''(r)}{w(r)}(\varphi(x)) \;\; \forall x \in P \quad ,$$

that the warping function w satisfies

$$w' \geq 0 \quad ,$$

that the model space M_w^m is balanced from below, and that P has finite $w-$flux. Then P is hyperbolic (parabolic) if and only if M_w^m is hyperbolic (parabolic).

3 Preliminaires

We assume throughout the paper that $\varphi : P^m \longrightarrow N^n$ is an isometric immersion of a complete non-compact Riemannian m-manifold P^m into a complete Riemannian manifold N^n with a pole $o \in N^n$. Recall that a pole is a point o such that the exponential map

$$\exp_o \colon T_o N^n \to N^n$$

is a diffeomorphism.

For every $x \in N^n - \{o\}$ we define $r(x) = r_o(x) = \mathrm{dist}_N(o, x)$, since o is a pole this distance is realized by the length of a unique geodesic from o to x, which is the *radial geodesic from* o. We also denote by $r|_P$ or by r the composition $r \circ \varphi : P \to \mathbb{R}_+ \cup \{0\}$. This composition is called the *extrinsic distance function* from o in P^m.

With the extrinsic distance we can construct the *extrinsic ball* $D_R(o)$ of radius R centered at o as

$$D_R(o) := \{x \in P : r(\varphi(x)) < R\} \quad .$$

Since $\partial D_t(o) = \Sigma_t^r$, the flux of the extrinsic distance function r on P is

$$J_r(t) = \int_{\partial D_t} |\nabla^P r| d\rho \quad ,$$

where the gradients of r in N and $r|_P$ in P are denoted by $\nabla^N r$ and $\nabla^P r$, respectively. These two gradients have the following basic relation, by virtue of the identification, given any point $x \in P$, between the tangent vector fields $X \in T_x P$ and $\varphi_{*_x}(X) \in T_{\varphi(x)} N$

$$\nabla^N r = \nabla^P r + (\nabla^N r)^\perp, \tag{36}$$

where $(\nabla^N r)^\perp(\varphi(x)) = \nabla^\perp r(\varphi(x))$ is perpendicular to $T_x P$ for all $x \in P$.

We now present the curvature restrictions which constitute the geometric framework of the present study.

Definition 2 Let o be a point in a Riemannian manifold N and let $x \in N - \{o\}$. The sectional curvature $K_N(\sigma_x)$ of the two-plane $\sigma_x \in T_x N$ is then called a *o-radial sectional curvature* of N at x if σ_x contains the tangent vector to a minimal geodesic from o to x. We denote these curvatures by $K_{o,N}(\sigma_x)$.

3.1 Model spaces

Throughout this paper we shall assume that the ambient manifold N^n has its o-radial sectional curvatures $K_{o,N}(x)$ bounded from above by the expression $K_w(r(x)) = -w''(r(x))/w(r(x))$, which are precisely the radial sectional curvatures of the *w-model space* M_w^m we are going to define.

Definition 3 (See [25, 12, 11]) A $w-$model M_w^m is a smooth warped product with base $B^1 = [0, \Lambda[\subset \mathbb{R}$ (where $0 < \Lambda \leq \infty$), fiber $F^{m-1} = \mathbb{S}_1^{m-1}$ (*i.e.* the unit $(m-1)$-sphere with standard metric), and warping function $w\colon [0, \Lambda[\to \mathbb{R}_+ \cup \{0\}$, with $w(0) = 0$, $w'(0) = 1$, and $w(r) > 0$ for all $r > 0$. The point $o_w = \pi^{-1}(0)$, where π denotes the projection onto B^1, is called the *center point* of the model space. If $\Lambda = \infty$, then o_w is a pole of M_w^m.

Proposition 1 *The simply connected space forms $\mathbb{K}^m(b)$ of constant curvature b are $w-$models with warping functions*

$$w_b(r) = \begin{cases} \frac{1}{\sqrt{b}} \sin(\sqrt{b}\, r) & \text{if } b > 0 \\ \quad r & \text{if } b = 0 \\ \frac{1}{\sqrt{-b}} \sinh(\sqrt{-b}\, r) & \text{if } b < 0. \end{cases}$$

Note that for $b > 0$ the function $w_b(r)$ admits a smooth extension to $r = \pi/\sqrt{b}$.

Proposition 2 (See [25, 11, 12]) *Let M_w^m be a $w-$model space with warping function $w(r)$ and center o_w. The distance sphere of radius r and center o_w in M_w^m is the fiber $\pi^{-1}(r)$. This distance sphere has the constant mean curvature $\eta_w(r) = \frac{w'(r)}{w(r)}$. On the other hand, the o_w-radial sectional curvatures of M_w^m at every $x \in \pi^{-1}(r)$ (for $r > 0$) are all identical and determined by*

$$K_{o_w, M_w}(\sigma_x) = -\frac{w''(r)}{w(r)}.$$

Remark 5 The $w-$model spaces are completely determined via w by the mean curvatures of the spherical fibers S_r^w:

$$\eta_w(r) = w'(r)/w(r) \quad ,$$

by the volume of the fiber

$$\mathrm{Vol}(S_r^w) \; = V_0\, w^{m-1}(r) \quad ,$$

and by the volume of the corresponding ball, for which the fiber is the boundary

$$\mathrm{Vol}(B_r^w) \; = \; V_0 \int_0^r \; w^{m-1}(t)\, dt \quad .$$

Here V_0 denotes the volume of the unit sphere $S_1^{0,m-1}$, (we denote in general as $S_r^{b,m-1}$ the sphere of radius r in the real space form $\mathbb{K}^m(b)$) . The latter two functions define the isoperimetric quotient function as follows

$$q_w(r) \; = \; \mathrm{Vol}(B_r^w)/\,\mathrm{Vol}(S_r^w) \quad .$$

We observe moreover that the flux of the geodesic distance function r_o from the center to the model space is

$$J_r^w(R) = \int_{S_R^w} |\nabla r| d\sigma = \mathrm{Vol}(S_R^w) \quad .$$

Besides the already defined comparison controllers for the radial sectional curvatures of N^n, we shall need two further purely intrinsic conditions on the model spaces:

Definition 4 A given $w-$model space M^m_w is called balanced from below and balanced from above, respectively, if the following weighted isoperimetric conditions are satisfied:

$$\begin{aligned} &\text{Balance from below:} \quad q_w(r)\,\eta_w(r) \geq 1/m \quad \text{for all} \quad r \geq 0 \quad ; \\ &\text{Balance from above:} \quad q_w(r)\,\eta_w(r) \leq 1/(m-1) \quad \text{for all} \quad r \geq 0 \quad . \end{aligned}$$

A model space is called *totally balanced* if it is balanced both from below and from above.

3.2 Laplacian comparison for radial functions

Let us recall the expression of the Laplacian on model spaces for radial functions

Proposition 3 (See [25], [11] and [12]) *Let M^n_w be a model space, denote by $r : M^n_w - \{o_w\} \to \mathbb{R}^+$ the geodesic distance from the center o_w, and let $f : \mathbb{R} \to \mathbb{R}$ be a smooth function, then*

$$\Delta^{M^n_w}(f \circ r) = f'' \circ r + (n-1)\,(f' \cdot \eta_w) \circ r \quad . \tag{37}$$

Applying the Hessian comparison theorems given in [11] we can obtain (see [22] for instance)

Proposition 4 *Let $\varphi : P^m \to N^n$ be an immersion into a manifold N with a pole. Suppose the the radial sectional curvatures K_N of N are bounded from above by the radial sectional curvatures of a model space M^m_w as follows:*

$$K_N \leq -\frac{w''}{w} \circ r \quad . \tag{38}$$

Let $f : \mathbb{R} \to \mathbb{R}$ be a smooth function with $f' \geq 0$, and denote by $r : P \to \mathbb{R}^+$ the extrinsic distance function. Then

$$\begin{aligned} \Delta^P(f \circ r) \geq &|\nabla^P r|\,(f'' - f' \cdot \eta_w) \circ r \\ &+ m\,(f' \cdot \eta_w) \circ r + m\langle \nabla^N r, H_P\rangle f' \circ r \quad , \end{aligned} \tag{39}$$

where H_P denotes the mean curvature vector of P in N.

3.2.1 Capacity and the Mean Exit Time function on Model spaces

One key purpose of this paper is to compare the capacity of extrinsic annuli of an immersed minimal submanifold with the capacity in an adequate model space. In the model space we can obtain the value of the capacity directly:

Proposition 5 (See [13]) *Let M_w^n be a model space. Then*

$$\operatorname{Cap}(A_{\rho,R}^w) = \left(\int_\rho^R \frac{ds}{\operatorname{Vol}(S_s^w)}\right)^{-1} = V_n \left(\int_\rho^R \frac{ds}{w^{n-1}(s)}\right)^{-1} \quad . \tag{40}$$

We note that the radial function $\Psi : M_w^m \to \mathbb{R}$ given by

$$\Psi(p) := \Psi_{\rho,R}^w(r(p)) \quad , \tag{41}$$

being

$$\Psi_{\rho,R}^w(t) = \int_\rho^t \frac{\operatorname{Cap}(A_{\rho,R}^w)}{\operatorname{Vol}(S_s^w)} ds \quad , \tag{42}$$

is the solution to the Dirichlet problem given in 25 for the annular region $A_{\rho,R}^w$, namely

$$\begin{cases} \Delta^{M_w^m}\Psi = 0 \\ \Psi|_{S_\rho^w} = 0 \\ \Psi|_{S_R^w} = 1 \end{cases} \tag{43}$$

Another important tool in this paper is the comparison result for the mean exit time. Let now E_R^w denote the mean time of the first exit from B_R^w for a Brownnian particle starting at o_w. A remark due to Dynkin in [8] claims that E_R^w is the continuous solution to the following Poisson equation with Dirichlet boundary data,

$$\begin{aligned} \Delta^{M_w^n} E_R^w =& -1 \\ E_R^w|_{S_R^w} =& 0. \end{aligned} \tag{44}$$

Since the ball B_R^w has maximal isotropy at the center o_w, so we have that E_R^w only depends on the extrinsic distance r. Therefore, we will write $E_R^w = E_R^w(r)$ and

Proposition 6 (See [22]) *Let M_w^n be a model space of dimension n then*

$$E_R^w(r) = \int_r^R q_w(t)dt \quad . \tag{45}$$

4 Proof of the main theorems of §2

4.1 Proof of theorem 3

Since the mean exit time function E_R^w is a radial function, we can transplant it to P using the extrinsic distance, hence, we also denote as $E_R^w : P \to \mathbb{R}$ the function given by $E_R^w(x) = E_R^w(r(x))$. To compare the mean exit time function, we need the following comparison for the mean exit time

Proposition 7 *([22]) Let $\varphi : P^m \longrightarrow N^n$ be an isometric, proper and minimal immersion of a complete non-compact Riemannian m-manifold P^m into a complete Riemannian manifold N^n with a pole $o \in N$. Let us suppose that the $o-$radial sectional curvatures of N are bounded from above by*

$$K_{o,N}(\sigma_x) \leq -\frac{w''(r)}{w(r)}(\varphi(x)) \;\; \forall x \in P\,,$$

and that the model space M_w^m is balanced from below, then

$$\Delta^P E_R^w \leq -1 = \Delta^P E_R. \tag{46}$$

Applying now the divergence theorem to inequality (46) we obtain

$$\begin{aligned} -\operatorname{Vol}(D_R) = &\int_{D_R} \Delta^P E_R^P(r) d\mu \geq \int_{D_R} \Delta^P E_R^w(r) d\mu \\ = &\int_{\partial D_R} E_R^w(r)' \langle \nabla^P r, \nu \rangle d\sigma = -q_w(R) \int_{\partial D_R} \|\nabla^P r\| d\sigma \end{aligned} \tag{47}$$

Therefore,

$$\frac{\operatorname{Vol}(D_R)}{\operatorname{Vol}(B_R^w)} \leq \frac{J_r(R)}{\operatorname{Vol}(S_R^w)} = \frac{J_r(R)}{J_r^w(R)} \quad . \tag{48}$$

Observe that equality in inequality (48) implies equality in inequality (47) and therefore, in inequality (46). Taking, thus, into account that $E_R^P = E_R^w$ in $x \in \partial D_R$, $\Delta E_R^P = \Delta E_R^w$ in $x \in D_R$, and the maximum principle, we obtain that equality in (48) implies

$$E_R^P = E_R^w \quad , \tag{49}$$

for all $x \in D_R$.

In order to obtain the monotonicity of the quotient $\frac{\operatorname{Vol}(D_R)}{\operatorname{Vol}(B_R^w)}$, we note that by the co-area formula we get for almost every $R \in \mathbb{R}$:

$$\begin{aligned} \left(\ln \frac{\operatorname{Vol}(D_R)}{\operatorname{Vol}(B_R^w)}\right)' = &\frac{\int_{\partial D_R} \frac{1}{\|\nabla^P r\|} d\sigma}{\operatorname{Vol}(D_R)} - \frac{\operatorname{Vol}(S_R^w)}{\operatorname{Vol}(B_R^w)} \\ \geq &\frac{\int_{\partial D_R} \|\nabla^P r\| d\sigma}{\operatorname{Vol}(D_R)} - \frac{\operatorname{Vol}(S_R^w)}{\operatorname{Vol}(B_R^w)} \\ = &\frac{\operatorname{Vol}(S_R^w)}{\operatorname{Vol}(D_R)} \left(\frac{J_r(R)}{\operatorname{Vol}(S_R^w)} - \frac{\operatorname{Vol}(D_R)}{\operatorname{Vol}(B_R^w)} \right) \\ \geq &0 \quad . \end{aligned} \tag{50}$$

Hence $\frac{\text{Vol}(D_R)}{\text{Vol}(B^w_R)}$ is a monotone non-decreasing function. The monotonicity of the quotient $\frac{\text{Vol}(D_R)}{\text{Vol}(B^w_R)}$ was first proved in [22]. To prove that also $\frac{J_r(R)}{J^w_r(R)}$ is a monotone nondecreasing function we need the following lemma

Lemma 1

$$\text{div}\left(\frac{\nabla^P E^w_R(r)}{\text{Vol}(B^w_r)}\right) \leq 0 \quad . \tag{51}$$

Proof Taking into account the product rule for the divergence and the mean exit time comparison result

$$\begin{aligned} \text{div}\ \left(\frac{\nabla^P E^w_R(r)}{\text{Vol}(B^w_r)}\right) =& \frac{\Delta^P E^w_R(r)}{\text{Vol}(B^w_r)} - \frac{\text{Vol}(B^w_r)'}{\text{Vol}(B^w_r)^2}\langle\nabla^P r, \nabla^P E^w_R(r)\rangle \\ =& \frac{\Delta^P E^w_R(r)}{\text{Vol}(B^w_r)} - \frac{\text{Vol}(S^w_r)}{\text{Vol}(B^w_r)^2} E^w_R(r)'\|\nabla^P r\|^2 \\ \leq& \frac{-1}{\text{Vol}(B^w_r)} + \frac{\|\nabla^P r\|^2}{\text{Vol}(B^w_r)} \leq 0 \quad . \end{aligned} \tag{52}$$

□

Using now this lemma and the divergence theorem in the extrinsic annulus $A_{\rho,R}$ for $\rho < R$

$$\begin{aligned} 0 \geq& \int_{A_{\rho,R}} \text{div}\ \left(\frac{\nabla^P E^w_R(r)}{\text{Vol}(B^w_r)}\right) d\mu \\ =& \int_{\partial D_R} \frac{E^w_R(r)'\|\nabla^P r\|}{\text{Vol}(B^w_r)} d\sigma - \int_{\partial D_\rho} \frac{E^w_R(r)'\|\nabla^P r\|}{\text{Vol}(B^w_r)} d\sigma \\ =& - \frac{J_r(R)}{\text{Vol}(S^w_R)} + \frac{J_r(\rho)}{\text{Vol}(S^w_\rho)} \quad . \end{aligned} \tag{53}$$

Therefore,

$$\frac{J_r(R)}{J^w_r(R)} \geq \frac{J_r(\rho)}{J^w_r(\rho)} \quad , \tag{54}$$

for any $R > \rho$, and the theorem is proven.

4.2 Proof of theorem 4

The corresponding Dirichlet problem for the capacity of the extrinsic annulus $A_{\rho,R}$ is

$$\begin{cases} \Delta^P \Psi = 0 \\ \Psi|_{\partial D_\rho} = 0 \\ \Psi|_{\partial D_R} = 1 \end{cases} \tag{55}$$

Let us transplant the function $\Psi^w_{\rho,R}$ with the extrinsic distance function r:

$$\Psi^w(p) : A_{\rho,R} \to \mathbb{R}, \quad p \to \Psi^w(p) := \Psi^w_{\rho,R}(r(p)) \quad . \tag{56}$$

Then, applying proposition 4

$$\Delta^P \Psi^w \geq m\left(1-|\nabla^P r|\right)\left((\Psi^w_{\rho,R})' \cdot \eta_w\right) \circ r \quad . \tag{57}$$

Taking into account that $\eta_w \geq 0$

$$\Delta^P \Psi^w \geq 0 = \Delta^P \Psi \quad . \tag{58}$$

Since $\Delta^P\left(\Psi^w - \Psi\right) \geq 0$ and since $\Psi_{\partial A_{\rho,R}} = \Psi^w_{\partial A_{\rho,R}}$, we have by the Maximum Principle that $\Psi^w \leq \Psi$ on $A_{\rho,R}$, and, since $\Psi_{\partial D_\rho} = \Psi^w_{\partial D_\rho} = 0$, we obtain

$$|\nabla^P \Psi^w| \leq |\nabla^P \Psi| \quad \text{ on } \partial D_\rho \quad . \tag{59}$$

Finally, we can estimate the capacity

$$\begin{aligned}
\mathrm{Cap}(A_{\rho,R}) &= \int_{\partial D_\rho} |\nabla^P \Psi| d\sigma \\
&\geq \int_{\partial D_\rho} |\nabla^P \Psi^w| d\sigma \\
&= (\Psi^w(\rho))' \int_{\partial D_\rho} |\nabla^P r| d\sigma \\
&= \mathrm{Cap}(A^w_{\rho,R}) \frac{J_r(\rho)}{J^w_r(\rho)} \quad ,
\end{aligned} \tag{60}$$

and the theorem follows.

4.3 Proof of theorem 5

With the flux we can provide an upper bound for the capacity (see inequality (28)). Using theorem 3 we obtain that

$$\begin{aligned}
\mathrm{Cap}(A_{\rho,R}) \leq & \frac{1}{\int_\rho^R \frac{ds}{\int_{\partial D_s} \|\nabla^P r\| d\sigma}} = \frac{1}{\int_\rho^R \frac{ds}{\frac{J_r(s)}{\mathrm{Vol}(S^w_s)} \mathrm{Vol}(S^w_s)}} \\
\leq & \frac{\frac{J_r(R)}{\mathrm{Vol}(S^w_R)}}{\int_\rho^R \frac{ds}{\mathrm{Vol}(S^w_s)}} = \frac{J_r(R)}{\mathrm{Vol}(S^w_R)} \mathrm{Cap}(A^w_{\rho,R}).
\end{aligned} \tag{61}$$

For the bounds from below, see [20]. Observe moreover that equality in the above inequality implies that

$$\int_t^R \left(\frac{\frac{J_r(R)}{\mathrm{Vol}(S^w_R)}}{\frac{J_r(s)}{\mathrm{Vol}(S^w_s)}} - 1 \right) \frac{1}{\mathrm{Vol}(S^w_s)} ds = 0. \tag{62}$$

Therefore

$$\frac{J_r(R)}{\mathrm{Vol}(S^w_R)} = \frac{J_r(s)}{\mathrm{Vol}(S^w_s)}, \tag{63}$$

for any $s \in [\rho, R]$. Then, by inequality (53)

$$\operatorname{div}\left(\frac{\nabla^P E_R^w(r)}{\operatorname{Vol}(B_r^w)}\right) = 0, \tag{64}$$

for any $p \in A_{\rho,R}$. From inequality (52)

$$\|\nabla^P r\| = 1, \tag{65}$$

for any $p \in A_{\rho,R}$, and hence, D_R is a minimal cone.

5 Proof of Theorem 1 and Corollary 1

This proof mimics the argument given in [27, Theorem 2], so we merely give a sketch emphasizing the points where the line of reasoning from [27] is modified to hold in the present more general setting.

First of all, note that we can construct the following order-preserving bijection

$$F : \mathbb{R}^+ \to \mathbb{R}^+, \quad F(t) = \int_0^t w(s)ds \quad .$$

Since $\varphi : P^m \to N^n$ is a complete proper and minimal immersion within a manifold with a pole N^n, applying proposition 4 we have

$$\Delta^P F \circ r \geq m w' \circ r \quad . \tag{66}$$

Hence, by using the assumption $w' > 0$, the extrinsic distance has no local maximum. Therefore for any R, $P^m \setminus D_R$ has no bounded components, being each component of $P^m \setminus D_R$ non compact, and the number of ends $\mathcal{E}_{D_R}(P)$ with respect to D_R is the number of connected components of $P^m \setminus D_R$.

Let us denote by $\left\{\Omega^i\right\}_{i=1}^{\mathcal{E}_{D_R}(P)}$ the set of $\mathcal{E}_{D_R}(P)$ connected components of $P^m \setminus D_R$ (every one of them is a minimal submanifold with boundary). Now we need the following lemma

Lemma 2 *For any connected component Ω_i of $P^m \setminus D_R$ the volume of the set*

$$D_t^{\Omega_i} = D_t \cap \Omega_i \quad ,$$

for any $t > R$, is bounded from below by

$$\operatorname{Vol}(D_t^{\Omega_i}) \geq V_m \left(\frac{t-R}{2}\right)^m \quad . \tag{67}$$

Proof Now pick a point $o' \in D_t^{\Omega_i}$ such that its extrinsic distance is $r_o(o') = \frac{R+t}{2}$, then the extrinsic ball $D_{\frac{t-R}{2}}^{\Omega_i}(o')$ in Ω_i centered at o' with radius $\frac{t-R}{2}$ satisfies

$$D_{\frac{t-R}{2}}^{\Omega_i}(o') \subset D_t^{\Omega_i} \quad . \tag{68}$$

Hence,

$$\mathrm{Vol}(D_t^{\Omega_i}) \geq \mathrm{Vol}(D_{\frac{t-R}{2}}^{\Omega_i}(o')) \quad . \tag{69}$$

Since $r(o') > R$ and the sectional curvatures of any tangent $2-$plane of the tangent space at every point in the geodesic ball $B^N_{\frac{t-R_0}{2}}(\varphi(o'))$ of the ambient manifold are non-positive, we can make use of the behavior of the volume quotient (claim (2) in theorem 3) for extrinsic balls to the immersion $\varphi : D^{\Omega_i}_{\frac{t-R_0}{2}}(o') \to B^N_{\frac{t-R_0}{2}}(\varphi(o'))$ with the new model comparison $w(r) = w_0(r) = r$ (namely $M^m_w = \mathbb{R}^m$) taking into account the asymptotic expansion for the volume of an extrinsic ball in a submanifold of an arbitrary Riemannian manifold obtained in [18],

$$\frac{\mathrm{Vol}(D^{\Omega_i}_{\frac{t-R}{2}}(o'))}{V_m\left(\frac{t-R}{2}\right)^m} \geq \lim_{s\to 0} \frac{\mathrm{Vol}(D^{\Omega_i}_s(o'))}{V_m s^m} \geq 1 \quad . \tag{70}$$

And the lemma is proved. □

Summing now in inequality (67) we obtain

$$\mathrm{Vol}(A_{R,t}) = \sum_{i=1}^{\mathcal{E}_{D_R}(P)} \mathrm{Vol}(D_t^{\Omega_i}) \geq \mathcal{E}_{D_R}(P) V_m \left(\frac{t-R}{2}\right)^m \quad . \tag{71}$$

Taking into account that $\mathrm{Vol}(A_{R,t}) \leq \mathrm{Vol}(D_t)$ and dividing by $\mathrm{Vol}(B_t^w)$ we obtain

$$\frac{\mathrm{Vol}(D_t)}{\mathrm{Vol}(B_t^w)} \geq \mathcal{E}_{D_R}(P) V_m \frac{\left(\frac{t-R}{2}\right)^m}{\mathrm{Vol}(B_t^w)} \quad . \tag{72}$$

We can split the last quotient by division and multiplication by t^m

$$\frac{\mathrm{Vol}(D_t)}{\mathrm{Vol}(B_t^w)} \geq \mathcal{E}_{D_R}(P) V_m \left(\frac{1-R/t}{2}\right)^m \frac{t^m}{\mathrm{Vol}(B_t^w)} \quad . \tag{73}$$

Hence, finally, using the explicit expression for $\mathrm{Vol}(B_t^w)$ the theorem follows.

In order to prove corollary 1, note that by the maximum principle $\mathcal{E}^P_{D_R}$ is a non-decreasing function with respect to R. By inequality (5) and the assumptions of the corollary we can conclude that $\mathcal{E}^P_{D_R}$ is stabilized, *i.e.* $\mathcal{E}^P_{D_R} =$ constant for sufficient large R.

Now let $F \subset P$ be an arbitrary compact subset. Using again the maximum principle of the immersion, we conclude that $\mathcal{E}_F(P)$ is a non-decreasing function of the compact set F (namely, if $F_1 \subset F_2$ then $\mathcal{E}_{F_1}(P) \leq \mathcal{E}_{F_2}(P)$). Taking into account that for any compact set K there exists R_K such that $K \subset D_{R_K}$, we finally obtain

$$\mathcal{E}(P) = \lim_{R\to\infty} \mathcal{E}_{D_R}(P) \quad , \tag{74}$$

and the corollary follows.

6 Corollaries and application of the extrinsic comparison theory

6.1 Relation between $w-$volume and $w-$flux of submanifolds

Under the hypothesis of theorem 3, if the submanifold has finite w-flux, the submanifold has finite w-volume. But in particular settings we can also state a reverse:

Theorem 6 *Let $\varphi : P^m \longrightarrow N^n$ be an isometric, proper, and minimal immersion of a complete non-compact Riemannian m-manifold P^m into a complete Riemannian manifold N^n with a pole $o \in N$. Let us suppose that the $o-$radial sectional curvatures of N are bounded from above by*

$$K_{o,N}(\sigma_x) \leq -\frac{w''(r)}{w(r)}(\varphi(x)) \ \ \forall x \in P\,.$$

Suppose that the model space M^m_w is balanced from below with warping function satisfying

$$w'(r) \geq 0 \quad \forall r \in \mathbb{R}_+.$$

Then, if the submanifold has finite w-volume, we have:

1. *The submanifold has finite w-flux.*
2. *$\mathrm{Flux}_w(P) = \mathrm{Vol}_w(P)$.*

Proof To prove the theorem let us state the following metric property for geodesic balls and geodesic spheres in a rotationally symmetric model space

Lemma 3 *Let M^m_w be a model space with*

$$w'(r) \geq 0 \quad \forall r \in \mathbb{R}_+.$$

Then

$$q_w(s) = \frac{\mathrm{Vol}(B^w_s)}{\mathrm{Vol}(S^w_s)} \leq s.$$

Proof Observe that

$$q_w(0) = 0, \tag{75}$$

and, since $w' \geq 0$,

$$q'_w(t) \leq 1, \forall t \geq 0. \tag{76}$$

Hence, by integrating the above inequality, the lemma follows. □

For the sake of completeness we also need to prove the following technical lemma

Lemma 4 *Let $f:\mathbb{R}\to\mathbb{R}$ be a non negative function ($f(x)\geq 0$), suppose for some $a>0$*

$$\int_a^\infty f(x)dx<\infty \quad . \tag{77}$$

Then, for any positive $\epsilon>0$, there exists a sequence $\{t_i\}_{i=1}^\infty$ with $t_i\to\infty$ when $i\to\infty$ such that

$$f(t_i)<\frac{\epsilon}{t_i} \quad ,$$

for any $i\geq 1$.

Proof For any given $\epsilon>0$, $t_1>a$ must exists such that

$$f(t_1)<\frac{\epsilon}{t_1}$$

because otherwise,

$$\int_a^\infty f(x)dx\geq\int_a^\infty\frac{\epsilon}{x}dx=\infty.$$

Similarly, $t_2>t_1$ must exists such that

$$f(t_2)<\frac{\epsilon}{t_2}$$

because otherwise,

$$\int_a^\infty f(x)dx\geq\int_{t_1}^\infty f(x)dx\geq\int_{t_1}^\infty\frac{\epsilon}{x}dx=\infty.$$

Since the same argument can be used iteratively the lemma is proved. □

Now, since P has finite w-volume, then there exists $S\in\mathbb{R}^+$ such that

$$\frac{\operatorname{Vol}(D_R)}{\operatorname{Vol}(B^w_R)}\leq\lim_{t\to\infty}\frac{\operatorname{Vol}(D_t)}{\operatorname{Vol}(B^w_t)}=S<\infty \quad . \tag{78}$$

By inequality (50)

$$\left(\ln\frac{\operatorname{Vol}(D_s)}{\operatorname{Vol}(B^w_s)}\right)'\geq\frac{\operatorname{Vol}(S^w_s)}{S\operatorname{Vol}(B^w_s)}\left(\frac{J_r(s)}{J^w_r(s)}-\frac{\operatorname{Vol}(D_s)}{\operatorname{Vol}(B^w_s)}\right)\geq 0 \quad . \tag{79}$$

Therefore, taking lemma 3 into account we get for almost every $s\in\mathbb{R}^+$:

$$0\leq\left(\frac{J_r(s)}{J^w_r(s)}-\frac{\operatorname{Vol}(D_s)}{\operatorname{Vol}(B^w_s)}\right)\leq S\left(\ln\frac{\operatorname{Vol}(D_s)}{\operatorname{Vol}(B^w_s)}\right)'s \quad . \tag{80}$$

But since $\operatorname{Vol}(D_t)$ is a non-decreasing function on t,

$$\begin{aligned}\infty>\ln S&=\lim_{t\to\infty}\ln\left(\frac{\operatorname{Vol}(D_t)}{\operatorname{Vol}(B^w_t)}\right)\\ &\geq\lim_{t\to\infty}\int_0^t\left(\ln\frac{\operatorname{Vol}(D_s)}{\operatorname{Vol}(B^w_s)}\right)'ds+\lim_{t\to 0}\ln\left(\frac{\operatorname{Vol}(D_t)}{\operatorname{Vol}(B^w_t)}\right)\\ &\geq\lim_{t\to\infty}\int_0^t\left(\ln\frac{\operatorname{Vol}(D_s)}{\operatorname{Vol}(B^w_s)}\right)'ds \quad .\end{aligned} \tag{81}$$

then, by using lemma 4 with $\epsilon = \frac{\varepsilon}{S}$, there exists a sequence $\{t_i\}_{i=1}^{\infty}$ with $t_i \to \infty$ when $i \to \infty$, and R_ε, such that

$$\left(\ln \frac{\operatorname{Vol}(D_{t_i})}{\operatorname{Vol}(B^w_{t_i})}\right)' t_i < \frac{\varepsilon}{S} \quad . \tag{82}$$

This holds for any $t_i > R_\varepsilon$. Applying inequality (80) taking into account the monotonicity of the flux and volume comparison quotients

$$0 \leq \operatorname{Flux}_w(P) - \operatorname{Vol}_w(P) \leq \varepsilon \quad , \tag{83}$$

for any $\varepsilon > 0$. Letting ε tend to 0, the theorem is proven. □

Remark 6 Observe, that in inequality (81) we have used

$$\begin{aligned}
\ln \frac{\operatorname{Vol}(D_{R_2})}{\operatorname{Vol}(B^w_{R_2})} &= \ln \operatorname{Vol}(D_{R_2}) - \int_{R_1}^{R_2} (\ln \operatorname{Vol}(B^w_s))'\, ds + \ln \operatorname{Vol}(B^w_{R_1}) \\
&\geq \int_{R_1}^{R_2} \left[(\ln \operatorname{Vol}(D_s))' - (\ln \operatorname{Vol}(B^w_s))'\right] ds + \ln \frac{\operatorname{Vol}(D_{R_1})}{\operatorname{Vol}(B^w_{R_1})} \\
&= \int_{R_1}^{R_2} \left(\ln \frac{\operatorname{Vol}(D_s)}{\operatorname{Vol}(B^w_s)}\right)' ds + \ln \frac{\operatorname{Vol}(D_{R_1})}{\operatorname{Vol}(B^w_{R_1})} \quad .
\end{aligned} \tag{84}$$

Since by the monotonicity of $\ln(\operatorname{Vol}(D_R))$ on R,

$$\ln \operatorname{Vol}(D_{R_2}) - \ln \operatorname{Vol}(D_{R_1}) \geq \int_{R_1}^{R_2} (\ln \operatorname{Vol}(D_s))'\, ds \quad . \tag{85}$$

Note that we can not make use of equality in the above inequality (85) and hence in inequality (84) because we do not know if the function $R \to \ln \operatorname{Vol}(D_R)$ is absolutely continuous. Since the extrinsic distance function r is a proper C^∞ function on $P \setminus \{o\}$, the set of critical values of r, by using the Sard's theorem, is a null set of R^+ and the function $R \to \operatorname{Vol}(D_R)$ (and so the function $R \to \ln \operatorname{Vol}(D_R)$) is smooth almost everywhere in $R \in \mathbb{R}^+$ (see theorem 5.8 of [26]), and obviously a non-decreasing function on R and that is enough to state inequality (85). See section 2.2 of the very recent [19] for an estimate of the measure of the critical set for the extrinsic distance function on minimal submanifolds in Cartan-Hadamard ambient manifolds.

6.2 Intrinsic versions

In this subsection we consider the intrinsic versions of Theorems 3, 4 and 5 assuming that $P^m = N^n$. In this case, the extrinsic distance to the pole p becomes the intrinsic distance in N^n, hence, for all R the extrinsic domains D_R become the geodesic balls B^N_R of the ambient manifold N^n. Then, for all $x \in P$

$$\nabla^P r(x) = \nabla^N r(x).$$

As a consequence, $\|\nabla^P r\| = 1$.

From this intrinsic viewpoint, we have the following isoperimetric and volume comparison inequalities.

Theorem 7 *Let N^n denote a complete Riemannian manifold with a pole p. Suppose that the p-radial sectional curvatures of N^n are bounded from above by the p_w-radial sectional curvatures of a w-model space M_w^n. Assume that*

$$w' \geq 0 \quad . \tag{86}$$

Then the capacity of the intrinsic annulus $A_{\rho,R}$ is bounded from below by

$$\frac{\operatorname{Vol}(\partial B_\rho^N)}{\operatorname{Vol}(S_\rho^w)} \leq \frac{\operatorname{Cap}(A_{\rho,R})}{\operatorname{Cap}(A_{\rho,R}^w)}$$

And, furthermore, if M_w^n is hyperbolic, then N^n is also hyperbolic.

Theorem 8 *Let N^n denote a complete Riemannian manifold with a pole p. Suppose that the p-radial sectional curvatures of N^n are bounded from above by the p_w-radial sectional curvatures of a w-model space M_w^n. Assume that M_w^n is balanced from below. Then,*

1. *for all $R > 0$*
$$\frac{\operatorname{Vol}(B_R^N)}{\operatorname{Vol}(B_R^w)} \leq \frac{\operatorname{Vol}(\partial B_R^N)}{\operatorname{Vol}(S_R^w)} \quad . \tag{87}$$
2. *The functions $\frac{\operatorname{Vol}(B_R^N)}{\operatorname{Vol}(B_R^w)}$ and $\frac{\operatorname{Vol}(\partial B_R^N)}{\operatorname{Vol}(S_R^w)}$ are non-decreasing on R.*
3. *Denoting by $E_R^N(x)$ the mean exit time function for the geodesic ball B_R^N in N and denoting by E_R^w the mean exit time function in the $R-$ball B_R^w in the model space M_w^n. If equality holds in (87) for some fixed $R > 0$ then for any $x \in B_R^N$, $E_R^N(x) = E_R^w(r(x))$.*
4. *The capacity of the intrinsic annulus $A_{\rho,R}$ is bounded from above by*
$$\frac{\operatorname{Cap}(A_{\rho,R})}{\operatorname{Cap}(A_{\rho,R}^w)} \leq \frac{\operatorname{Vol}(\partial B_R^N)}{\operatorname{Vol}(S_R^w)}$$

Furthermore, if we suppose that there exists a finite real constant $C < \infty$ such that $\frac{\operatorname{Vol}(B_R^N)}{\operatorname{Vol}(B_R^w)} < C$ (or $\frac{\operatorname{Vol}(\partial B_R^N)}{\operatorname{Vol}(S_R^w)} < C$) then if M_w^n is parabolic, N is parabolic, and

$$\lim_{R\to\infty} \frac{\operatorname{Vol}(B_R^N)}{\operatorname{Vol}(B_R^w)} = \lim_{R\to\infty} \frac{\operatorname{Vol}(\partial B_R^N)}{\operatorname{Vol}(S_R^w)}.$$

6.3 Upper bounds for the fundamental tone

S.T. Yau suggested in [28] the "very interesting" question to find an upper estimate to the first Dirichlet eigenvalue of minimal surfaces.

Recall that for any precompact region $\Omega \subset M$ in a Riemannian manifold M, the first eigenvalue $\lambda_1(\Omega)$ of the Dirichlet problem in Ω for the Laplace operator is defined by the variational property

$$\lambda_1(\Omega) = \inf_u \frac{\int \|\nabla u\|^2 d\mu}{\int \|u\|^2} \tag{88}$$

where the inf is taken over all Lipschitz functions $u \neq 0$ compactly supported in Ω.

The fundamental tone $\lambda^*(M)$ of a complete Riemannian manifold can be obtained as the limit of the first Dirichlet eigenvalues of the precompact open sets in any exhaustion sequence $\{\Omega_n\}_{n\in\mathbb{N}}$ for M, see [12]

$$\lambda^*(M) = \lim_{n\to\infty} \lambda_1(\Omega_n) \quad . \tag{89}$$

In this section, we shall impose flux and volume restrictions not on the submanifold P but on one end V of the submanifold with respect to the extrinsic ball D_{R_0}. Let us denote D_R^V the intersection of the extrinsic ball D_R with the end V with respect to D_{R_0}

$$D_R^V = D_R \cap V \quad . \tag{90}$$

Let us denote $J_r^V(R)$ the flux of the extrinsic distance in the end V, namely

$$J_r^V(R) = \int_{\partial D_R \cap V} |\nabla^P r| d\sigma \quad . \tag{91}$$

With this setting we then have:

Theorem 9 *Let $\varphi : P^m \longrightarrow N^n$ be an isometric, proper and minimal immersion of a complete non-compact Riemannian m-manifold P^m into a complete Riemannian manifold N^n with a pole $o \in N$. Let us suppose that the $o-$radial sectional curvatures of N are bounded from above by*

$$K_{o,N}(\sigma_x) \leq -\frac{w''(r)}{w(r)}(\varphi(x)) \;\; \forall x \in P \,,$$

and that the model space M_w^m is balanced from below. Suppose moreover that there exists an end V with respect to an extrinsic ball D_{R_0} with finite w-flux. Then

$$\lambda^*(P) \leq \frac{Flux_w(V)}{\mathrm{Vol}_w(V)} \limsup_{t\to\infty} \left(\frac{1}{\mathrm{Vol}(B_t^w) \int_t^\infty \frac{ds}{\mathrm{Vol}(S_s^w)}} \right) \quad . \tag{92}$$

Proof Due to the relation between the first Dirichlet eigenvalue and the capacity given in [13] we can conclude for the extrinsic ball D_R^V that

$$\lambda_1(D_R^V) \leq \frac{\mathrm{Cap}(A_{t,R}^V)}{\mathrm{Vol}(D_t^V)} \quad . \tag{93}$$

Being $t < R$ and $A_{t,R}^V$ the extrinsic annulus in V. Hence, by the theorem 5

$$\lambda_1(D_R^V) \leq \frac{\frac{J_r^V(R)}{J_r^w(R)}}{\frac{\mathrm{Vol}(D_t^V)}{\mathrm{Vol}(B_t^w)}} \frac{\mathrm{Cap}(A_{t,R}^w)}{\mathrm{Vol}(B_t^w)} \quad . \tag{94}$$

For any $t < R$. Finally, taking into account that $\lambda(D_R) \leq \lambda_1(D_R^V)$ (by the monotonicity of the first eigenvalue), and letting R tend to infinity we have

$$\lambda^*(P) \leq \frac{\mathrm{Flux}_w(V)}{\frac{\mathrm{Vol}(D_t^V)}{\mathrm{Vol}(B_t^w)}} \frac{1}{\mathrm{Vol}(B_t^w)\int_t^\infty \frac{ds}{\mathrm{Vol}(S_s^w)}} \quad . \tag{95}$$

Taking limits, the theorem follows. □

Obviously, by using theorem 6 we also have the following:

Corollary 6 *Under the assumptions of theorem 9 suppose moreover*

$$w' \geq 0.$$

Then,

$$\lambda^*(P) \leq \limsup_{t\to\infty} \left(\frac{1}{\mathrm{Vol}(B_t^w)\int_t^\infty \frac{ds}{\mathrm{Vol}(S_s^w)}} \right) \quad . \tag{96}$$

Using the Cheeger isoperimetric constant we can deduce the following lower bounds

Theorem 10 *Let $\varphi : P^m \longrightarrow N^n$ be an isometric, proper and minimal immersion of a complete non-compact Riemannian m-manifold P^m into a complete Riemannian manifold N^n with a pole $o \in N^n$. Let us suppose that the $o-$radial sectional curvatures of N^n are bounded from above by*

$$K_{o,N}(\sigma_x) \leq -\frac{w''(r)}{w(r)}(\varphi(x)) \;\; \forall x \in P\,,$$

and that the model space M_w^m is balanced from below. Suppose moreover that

$$L := \sup_{t\in\mathbb{R}^+} q_w(t) < \infty.$$

Then

$$\frac{1}{4L^2} \leq \lambda^*(P) \quad . \tag{97}$$

Proof Consider $\Omega \subset P^m$ a smooth domain with smooth boundary $\partial\Omega$. Using the transplanted mean exit function in a similar way as in the proof of theorem 3 we obtain:

$$\begin{aligned} -\operatorname{Vol}(\Omega) &= \int_\Omega \Delta^P E_R d\mu \geq \int_\Omega \Delta^P E_R^w d\mu = \int_{\partial\Omega} E_R^w(r)' \langle \nabla^P r, \nu \rangle d\sigma \\ &\geq -\int_{\partial\Omega} q_w(r) \langle \nabla^P r, \nu \rangle d\sigma \geq -\int_{\partial\Omega} q_w(r) d\sigma \\ &\geq -L \operatorname{Vol}(\partial\Omega) \quad . \end{aligned} \tag{98}$$

Hence, for any $\Omega \subset P$,

$$\frac{\operatorname{Vol}(\partial\Omega)}{\operatorname{Vol}(\Omega)} \geq \frac{1}{L} \quad . \tag{99}$$

Thence the Cheeger constant $h(P)$ (see [3]) satisfies

$$h(P) \geq \frac{1}{L} \quad . \tag{100}$$

Taking into account that

$$\lambda^*(P) \geq \frac{1}{4} \left(h(P)\right)^2 \quad , \tag{101}$$

the theorem follows. □

As an immediate consequence of the previous theorems and corollaries in the particular setting of a minimal submanifold in a Cartan-Hadamard ambient manifold we have the following:

Corollary 7 *Let $\varphi : P^m \longrightarrow N^n$ be a complete minimal immersion into a simply connected Cartan-Hadamard manifold N^n with sectional curvatures $K_N \leq b \leq 0$. Suppose moreover that there exists an end V with respect to an extrinsic ball D_{R_0} with finite w_b-volume. Then*

$$\frac{-(m-1)^2 b}{4} \leq \lambda^*(P) \leq -(m-1)^2 b \quad . \tag{102}$$

Remark 7 See [23, 6, 2, 9] for upper and lower bounds for the fundamental tone of manifolds with bounded extrinsic curvature. Note that if $b = 0$ in the above theorem, $\lambda^*(P) = 0$.

6.4 Applications to minimal submanifolds in $\mathbb{R}^n$

If P^m is a minimal submanifold in $\mathbb{R}^n$, it is well known that the extrinsic distance r satisfies

$$\Delta^P r^2 = 2m \tag{103}$$

Applying the divergence theorem

$$
\begin{aligned}
2m\,\mathrm{Vol}(D_R) = & \int_{D_R} \Delta^p r^2 d\mu = \int_{\partial D_R} 2r\langle \nabla r, \nu\rangle d\sigma \\
=& 2R\int_{\partial D_R}\langle \nabla r, \frac{\nabla r}{|\nabla r|}\rangle d\sigma = 2R\int_{\partial D_R}|\nabla r| d\sigma \\
=& 2m\frac{\mathrm{Vol}(B_R^{w_0})}{\mathrm{Vol}(S_R^{w_0})}\int_{\partial D_R}|\nabla r| d\sigma
\end{aligned} \tag{104}
$$

hence, the volume comparison quotient $\frac{\mathrm{Vol}(D_R)}{\mathrm{Vol}(B_R^{w_0})}$ is just

$$
\frac{\mathrm{Vol}(D_R)}{\mathrm{Vol}(B_R^{w_0})} = \frac{J_r(R)}{J_r^{w_0}(R)} \quad . \tag{105}
$$

And therefore, we can state that

Corollary 8 *Let P^m be a minimal submanifold properly immersed in the Euclidean space $\mathbb{R}^n$. Then*

$$
E_R^P(x) = E_R^{\mathbb{R}^m}(r(x)) \quad ,
$$

where $E_R^P(x)$ denotes the mean exit time from D_R for a Brownian particle starting at $x \in D_R$, and $E_R^w(r)$ denotes the (rotationally symmetric) mean exit time function for the $R-$ball B_R^w in the model space M_w^m

If we have finite w_0-volume ($\sup_{R\in\mathbb{R}^+}\frac{\mathrm{Vol}(D_R)}{\mathrm{Vol}(B_R^{w_0})} < \infty$) we also get:

Corollary 9 *Let P^m be a minimal submanifold immersed in $\mathbb{R}^n$, suppose moreover that P has finite w_0-volume then:*

1. *P is parabolic if $m = 2$ and if $m \geq 3$, P is hyperbolic.*
2. *$\lambda^*(P) = 0$.*

On the other hand, in special geometric settings the finiteness of the w_0-volume is related to the number of ends

Theorem B *(See [1] and [5]) Let P^m be a minimal submanifold properly immersed in $\mathbb{R}^n$ with finite total scalar curvature* i.e. $\int_P \|B^P\|^m d\mu < \infty$ *where $\|B^P\|$ denotes the norm of the second fundamental form in P, then*

$$
\frac{J_r(R)}{J_r^{w_0}(R)} \leq \mathcal{E}(P), \tag{106}
$$

provided either of the following two conditions hold

1. *$m = 2$, $n = 3$ and each end of P is embedded.*
2. *$m \geq 3$.*

Where $\mathcal{E}(P)$ denotes the finite number of ends of P.

This relation between the number of ends and the flux quotient allow us to state

Corollary 10 *Let P^m be a minimal submanifold properly immersed in $\mathbb{R}^n$ with finite total scalar curvature and either $m \geq 3$, or $m = 2$ $n = 3$ and each end of P is embedded, then for any $\rho > 0$ and any $R > \rho$*

$$1 \leq \frac{\mathrm{Cap}(A_{\rho,R})}{\mathrm{Cap}(A^w_{\rho,R})} \leq \mathcal{E}(P) \quad . \tag{107}$$

And for the fundamental tone

$$\lambda^*(P) = 0 \quad . \tag{108}$$

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